Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number	Candidate Number

Monday 13 May 2019

Afternoon (Time: 1 hour 40 minutes) | Paper Reference **8FM0-01**

Further Mathematics

Advanced Subsidiary
Paper 1: Core Pure Mathematics

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 80.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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1/1/1/1/1

P 5 8 3 3 3 A 0 1 3 2

Pearson

1.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix **M** is non-singular.

(2)

The transformation T of the plane is represented by the matrix **M**.

The triangle R is transformed to the triangle S by the transformation T.

Given that the area of S is 63 square units,

(b) find the area of R.

(2)

(c) Show that the line $y = 2x$ is invariant under the transformation T.

(2)

1.a) For a matrix to be non-singular, the determinant $\neq 0$

$$\boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |A| = \det A = ad - bc}$$

$$M = \begin{bmatrix} 4 & -5 \\ 2 & -7 \end{bmatrix} \qquad \det M = 4(-7) - 2(-5)$$

$$= -18 \neq 0 \quad \therefore M \text{ is non-singular}$$

b) $\underset{63}{S} = \underset{18}{M}(\underset{A}{R})$

The determinant of a matrix transformation gives the scale factor

$\therefore$ scale factor of transformation $T = 18$

negative sign shows that the orientation is reversed

$$\therefore A_R = \frac{63}{18} = \frac{7}{2}$$

c) $$\begin{bmatrix} 4 & -5 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} x \\ 2x \end{bmatrix} = \begin{bmatrix} 4x - 5(2x) \\ 2x - 7(2x) \end{bmatrix} = \begin{bmatrix} -6x \\ -12x \end{bmatrix}$$

P 5 8 3 3 3 A 0 2 3 2

Question 1 continued

If line $y = 2x$ is invariant, the point after the transformation should also lie on $y = 2x$

check $(-6x, -12x)$ $y = 2(-6x) = -12x$

$\therefore$ $y = 2x$ is invariant

(Total for Question 1 is 6 marks)

2. The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$, giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where p, q, r and s are integers to be found.

(5)

2. METHOD 1 (using substitution):

$w = x + 3$

$x = w - 3$

$$2x^3 + 6x^2 - 3x + 12 = 0$$

$$2(w-3)^3 + 6(w-3)^2 - 3(w-3) + 12 = 0$$

$$2(w^3 - 9w^2 + 27w - 27) + 6(w^2 - 6w + 9) - 3(w-3) + 12 = 0$$

$$2w^3 - 12w^2 + 15w + 21 = 0$$

METHOD 2 (using sum/product rules):

$$\boxed{\begin{array}{l} ax^3 + bx^2 + cx + d = 0 \\ \alpha + \beta + \gamma = -\dfrac{b}{a} \quad \alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{c}{a} \quad \alpha\beta\gamma = \dfrac{-d}{a} \end{array}}$$

$$2x^3 + 6x^2 - 3x + 12 = 0$$

$\alpha + \beta + \gamma = -\dfrac{b}{a} = -\dfrac{(6)}{2} = -3$ $\quad \alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{c}{a} = \dfrac{-3}{2}$ $\quad \alpha\beta\gamma = -\dfrac{d}{a} = -\dfrac{12}{2} = -6$

new roots: $\underset{p}{(\alpha + 3)}, \underset{q}{(\beta + 3)}, \underset{r}{(\gamma + 3)}$ $\quad$ new equation $\rightarrow aw^3 + bw^2 + cw + d = 0$

$$-\frac{b}{a} = p + q + r = \alpha + 3 + \beta + 3 + \gamma + 3$$
$$= (\alpha + \beta + \gamma) + 9$$
$$= (-3) + 9 = 6 \qquad b = -6$$

Question 2 continued

$\frac{c}{a} = pq + pr + qr = (\alpha+3)(\beta+3) + (\alpha+3)(\gamma+3) + (\beta+3)(\gamma+3)$

$= \alpha\beta + 3\alpha + 3\beta + 9 + \alpha\gamma + 3\alpha + 3\gamma + 9 + \beta\gamma + 3\beta + 3\gamma + 9$

$= (\alpha\beta + \alpha\gamma + \beta\gamma) + 6(\alpha + \beta + \gamma) + 27$

$= \left(-\frac{3}{2}\right) + 6(-3) + 27 = \frac{15}{2}$ $\quad c = \frac{15}{2}$

$\frac{-d}{a} = pqr = (\alpha+3)(\beta+3)(\gamma+3)$

$= (\alpha\beta + 3\alpha + 3\beta + 9)(\gamma+3)$

$= \alpha\beta\gamma + 3\alpha\beta + 3\alpha\gamma + 9\alpha + 3\beta\gamma + 9\beta + 9\gamma + 27$

$= (\alpha\beta\gamma) + 3(\alpha\beta + \alpha\gamma + \beta\gamma) + 9(\alpha + \beta + \gamma) + 27$

$= (-6) + 3\left(-\frac{3}{2}\right) + 9(-3) + 27 = -\frac{21}{2}$ $\quad d = \frac{21}{2}$

$w^3 - 6w^2 + \frac{15}{2}w - \frac{21}{2} = 0$

$2w^3 - 12w^2 + 15w + 21 = 0$

(Total for Question 2 is 5 marks)

3. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

(6)

INDUCTION :
- Prove true for base case
- Assume true for $n=k$
- Consider $n=k+1$ & replace by assumption
- Conclusion

Base case : $n=1$

LHS : $\sum_{r=1}^{1} \frac{1}{(2r-1)(2r+1)} = \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{3}$

RHS : $\frac{1}{2(1)+1} = \frac{1}{3}$

RHS = LHS
$\therefore$ statement true for $n=1$

Assume true for $n=k$

$$\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$$

Consider $n=k+1$

$$\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{(2k+3)k+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$

$$= \frac{k+1}{2k+3} = \frac{k+1}{2(k+1)+1}$$

$\therefore$ hence true for $n=k+1$

Question 3 continued

Since statement is true for $n=1$, and we have proved if true for $n=k$, it is true for $n=k+1$, thus by mathematical induction, the result holds true for all positive integers

(Total for Question 3 is 6 marks)

4. The line l has equation

$$\frac{x+2}{1}=\frac{y-5}{-1}=\frac{z-4}{-3}$$

The plane Π has equation

$$\mathbf{r}.(\mathbf{i}-2\mathbf{j}+\mathbf{k})=-7$$

Determine whether the line l intersects Π at a single point, or lies in Π, or is parallel to Π without intersecting it.

(5)

4. $l: \dfrac{x-(-2)}{1} = \dfrac{y-(5)}{-1} = \dfrac{z-(4)}{-3}$

$l: \mathbf{r} = \begin{pmatrix}-2\\5\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\-3\end{pmatrix}$

$\Pi: \mathbf{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = -7$

substitute equation for l into equation for plane

$\begin{pmatrix}-2+\lambda\\5-\lambda\\4-3\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = (-2+\lambda)-2(5-\lambda)+(4-3\lambda)$

$= -2+\lambda-10+2\lambda+4-3\lambda = -8$

However Π equation states $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = -7$

$-8 \neq -7$

$\therefore$ l must be parallel to Π (i.e not in it)

* If the solutions were infinite & consistent, the line would lie on the plane

↳ If there was just 1 solution for λ, it would give the point of intersection between l & Π

5.

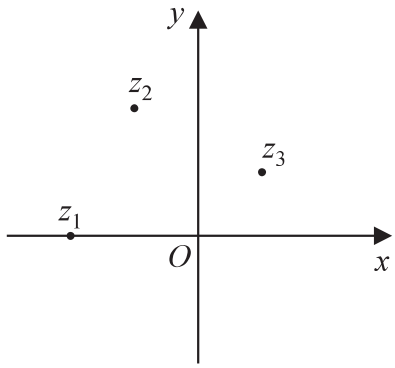


Figure 1

The complex numbers $z_1 = -2$, $z_2 = -1 + 2\text{i}$ and $z_3 = 1 + \text{i}$ are plotted in Figure 1, on an Argand diagram for the complex plane with $z = x + \text{i}y$

(a) Explain why z_1, z_2 and z_3 cannot all be roots of a quartic polynomial equation with real coefficients.

(2)

(b) Show that $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_1}\right) = \dfrac{\pi}{4}$

(3)

(c) Hence show that $\arctan(2) - \arctan\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$

(2)

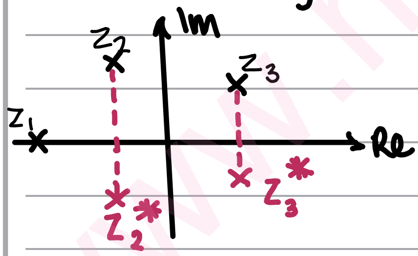
A copy of Figure 1, labelled Diagram 1, is given on page 12.

(d) Shade, on Diagram 1, the set of points of the complex plane that satisfy the inequality

$$|z + 2| \leqslant |z - 1 - \text{i}|$$

(2)

5. a) Because complex roots of a real polynomial occur in conjugate pairs



There are already at least 5 roots
$z_1, z_2, z_2^*, z_3, z_3^*$

but a quartic has max. 4 roots

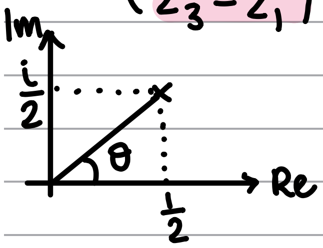
$\therefore$ not a quartic

b) $z_1 = -2 \qquad z_2 = -1 + 2i \qquad z_3 = 1 + i$

$z_2 - z_1 = 1 + 2i$
$z_3 - z_1 = 3 + i$

Question 5 continued

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(\frac{1+2i}{3+i}\right) = \arg\left(\frac{1+i}{2}\right)$$



$$\arg\left(\frac{1+i}{2}\right) = \tan^{-1}\left(\frac{1/2}{1/2}\right) = \frac{\pi}{4}$$

c) $\arg\left(\frac{a}{b}\right) = \arg(a) - \arg(b)$

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_1)$$

$$= \arg(1+2i) - \arg(3+i)$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{2}{1}\right) - \tan^{-1}\left(\frac{1}{3}\right)$$

$$\frac{\pi}{4} = \arctan(2) - \arctan\left(\frac{1}{3}\right)$$

d) $|z-(-2)| \leqslant |z-(1+i)|$

$\therefore$ loci represent points closer to -2 so on the left side of the perpendicular bisector to the line connecting $(1+i)$ & (-2)

Question 5 continued

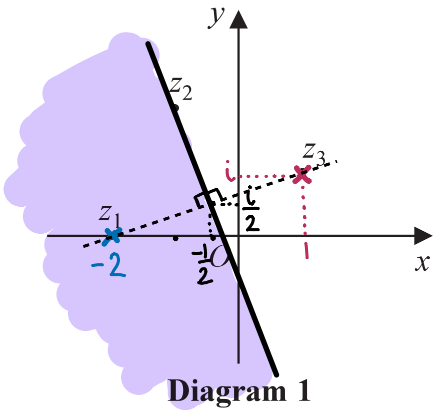


Diagram 1

6. An art display consists of an arrangement of n marbles.

When arranged in ascending order of mass, the mass of the first marble is 10 grams. The mass of each subsequent marble is 3 grams more than the mass of the previous one, so that the rth marble has mass $(7 + 3r)$ grams.

(a) Show that the mean mass, in grams, of the marbles in the display is given by

$$\frac{1}{2}(3n+17)$$

(3)

Given that there are 85 marbles in the display,

(b) use the standard summation formulae to find the standard deviation of the mass of the marbles in the display, giving your answer, in grams, to one decimal place.

(6)

6. a) 10g, $(10+3)g = 13g$, $(13+3)g = 16g$, . . . rth $(7+3r)g$

mean = $\dfrac{\text{sum of all marbles' mass}}{\text{total no. of marbles}}$

$$\text{mean} = \frac{1}{n} \times \sum_{r=1}^{n}(7+3r)$$

$$= \frac{1}{n}\left(7\sum_{r=1}^{n} + 3\sum_{r=1}^{n} r\right)$$

triangular number formula $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$$= \frac{1}{n}\left(7n + 3\left(\frac{1}{2}n(n+1)\right)\right)$$

$$= \frac{1}{n}\left(7n + \frac{3}{2}n^2 + \frac{3}{2}n\right)$$

$$= \frac{17}{2} + \frac{3n}{2}$$

$$= \frac{1}{2}(3n+17)$$

b) Standard deviation $= \sqrt{\dfrac{\Sigma(x^2)}{n} - (\bar{x})^2}$

Question 6 continued

$$\sum_{r=1}^{n}(7+3r)^2 = \sum_{r=1}^{n}(49+42r+9r^2)$$

$$= 49\sum_{r=1}^{n}1 + 42\sum_{r=1}^{n}r + 9\sum_{r=1}^{n}r^2$$

USING STANDARD SUMMATIONS

$$\sum_{r=1}^{n}r^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \sum_{r=1}^{n}r \; \frac{1}{2}n(n+1)$$

from formula booklet

triangular no.s

$$= 49(n) + 42\left(\frac{1}{2}n(n+1)\right) + 9\left(\frac{1}{6}n(n+1)(2n+1)\right)$$

$$= 49n + 21n^2 + 21n + \frac{3}{2}\left(n(2n^2+3n+1)\right)$$

$$= 60n + 21n^2 + 3n^3 + \frac{9}{2}n^2 + \frac{3n}{2}$$

$$= 3n^3 + \frac{51}{2}n^2 + \frac{143}{2}n$$

when $n = 85$ $\quad \sum_{r=1}^{85}(7+3r)^2 = 3(85)^3 + \frac{51}{2}(85)^2 + \frac{143}{2}(85)$

$$= 2032690$$

standard deviation $= \sqrt{\frac{2032690}{85} - (\bar{x})^2}$

$$= \sqrt{23914 - 136^2}$$

$$= \sqrt{5418} = 73.6\text{ g}$$

(Total for Question 6 is 9 marks)

7. $$f(z) = z^3 - 8z^2 + pz - 24$$

where p is a real constant.

Given that the equation $f(z) = 0$ has distinct roots

$$\alpha,\ \beta \text{ and } \left(\alpha + \frac{12}{\alpha} - \beta\right)$$

(a) solve completely the equation $f(z) = 0$ **(6)**

(b) Hence find the value of p. **(2)**

7. a) $f(z) = z^3 - 8z^2 + pz - 24$

$$\boxed{\begin{array}{l} ax^3 + bx^2 + cx + d = 0 \\ \alpha+\beta+\gamma = -\frac{b}{a} \quad \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} \quad \alpha\beta\gamma = \frac{-d}{a} \end{array}}$$

given that $f(z)$ has roots $\alpha, \beta, \left(\alpha + \frac{12}{\alpha} - \beta\right)$

sum of roots $= -\frac{b}{a} = -\frac{(-8)}{1} = 8 = \alpha + \cancel{\beta} + \alpha + \frac{12}{\alpha} - \cancel{\beta}$

$$8 = 2\alpha + \frac{12}{\alpha}$$

$$2\alpha^2 - 8\alpha + 12 = 0$$

$$\alpha^2 - 4\alpha + 6 = 0$$

$$\alpha = 2 \pm \sqrt{2}i$$

$\therefore 8 = 2 + \cancel{\sqrt{2}i} + 2 - \cancel{\sqrt{2}i} + (3^{rd} \text{ root})$

3^{rd} root $= 4$

hence roots of $f(z)$ are $4, 2+i\sqrt{2}, 2-i\sqrt{2}$

Question 7 continued

b) $\frac{c}{a} = \frac{p}{1} = p = (2+i\sqrt{2})(4) + (2-i\sqrt{2})(4) + (2+i\sqrt{2})(2-i\sqrt{2})$

$p = 8 + \cancel{4\sqrt{2}i} + 8 - \cancel{4\sqrt{2}i} + 6$

$p = 22$

8. A gas company maintains a straight pipeline that passes under a mountain.

The pipeline is modelled as a straight line and one side of the mountain is modelled as a plane.

There are accessways from a control centre to two access points on the pipeline.

Modelling the control centre as the origin O, the two access points on the pipeline have coordinates $P(-300, 400, -150)$ and $Q(300, 300, -50)$, where the units are metres.

(a) Find a vector equation for the line PQ, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is a scalar parameter.

(2)

The equation of the plane modelling the side of the mountain is $2x + 3y - 5z = 300$

The company wants to create a new accessway from this side of the mountain to the pipeline.

The accessway will consist of a tunnel of shortest possible length between the pipeline and the point $M(100, k, 100)$ on this side of the mountain, where k is a constant.

(b) Using the model, find

(i) the coordinates of the point at which this tunnel will meet the pipeline,

(ii) the length of this tunnel.

(7)

It is only practical to construct the new accessway if it will be significantly shorter than both of the existing accessways, OP and OQ.

(c) Determine whether the company should build the new accessway.

(2)

(d) Suggest one limitation of the model.

(1)

8.a) $P = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix}$ $\quad Q = \begin{pmatrix} 300 \\ 300 \\ -50 \end{pmatrix}$

$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{pmatrix} 300 \\ 300 \\ -50 \end{pmatrix} - \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} = \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}$

$\therefore PQ : r = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}$

Question 8 continued

b) (i) If M is on the mountain → M (100, k, 100)

$$2(100) + 3(k) - 5(100) = 300$$

$$k = 200$$

$$\therefore M = (100, 200, 100)$$

Let X be a point on the pipeline

$$X = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}$$

$$\overrightarrow{MX} = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} -400 + 600\lambda \\ 200 - 100\lambda \\ -250 + 100\lambda \end{pmatrix}$$

$\overrightarrow{MX}$ must be perpendicular to the pipeline

if a & b are perpendicular → $a \cdot b = 0$

$$\therefore \overrightarrow{MX} \cdot \overrightarrow{PQ} = \begin{pmatrix} -400 + 600\lambda \\ 200 - 100\lambda \\ -250 + 100\lambda \end{pmatrix} \cdot \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}$$

$$= 600(-400 + 600\lambda) - 100(200 - 100\lambda) + 100(-250 + 100\lambda)$$

$$= -285000 + 380000\lambda = 0$$

$$\lambda = \frac{3}{4}$$

so $X = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix} = \begin{pmatrix} 150 \\ 325 \\ -75 \end{pmatrix}$ $\therefore X = (150, 325, -75)$

(ii) Length of tunnel $= |MX| = \left| \begin{pmatrix} 150 \\ 325 \\ -75 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} \right|$

Question 8 continued

$$\boxed{\left|\begin{pmatrix}a\\b\\c\end{pmatrix}\right| = \sqrt{a^2+b^2+c^2}}$$

$$= \left|\begin{pmatrix}50\\125\\-175\end{pmatrix}\right|$$

$$= \sqrt{50^2+125^2+(-175)^2}$$

$$= 25\sqrt{78}\text{ m} \approx 221\text{ m}$$

c) $|\overrightarrow{OP}| = \left|\begin{pmatrix}-300\\400\\-150\end{pmatrix}\right| = \sqrt{(-300)^2+(400)^2+(-150)^2}$

$$= 50\sqrt{109} \approx 522\text{ m}$$

$|\overrightarrow{OQ}| = \left|\begin{pmatrix}300\\300\\-50\end{pmatrix}\right| = \sqrt{(300)^2+(300)^2+(-50)^2}$

$$= 50\sqrt{73} \approx 427\text{ m}$$

The length of the new tunnel is significantly shorter than both $|OP|$ & $|OQ|$

$\therefore$ likely that the company will decide to build the accessway

d) The mountainside is not likely to be a perfectly flat plane $\therefore$ so may not be a good model

9. $$f(x) = 2x^{\frac{1}{3}} + x^{-\frac{2}{3}} \qquad x > 0$$

The finite region bounded by the curve $y = f(x)$, the line $x = \frac{1}{8}$, the x-axis and the line $x = 8$ is rotated through θ radians about the x-axis to form a solid of revolution.

Given that the volume of the solid formed is $\frac{461}{2}$ units cubed, use algebraic integration to find the angle θ through which the region is rotated.

(8)

9. $f(x) = 2x^{\frac{1}{3}} + x^{-\frac{2}{3}}$

$$\boxed{V_{\substack{\text{about}\\x\text{-axis}}} = \pi \int_a^b y^2\, dx}$$

Volume (rotated fully) about x axis $= \pi \int_{\frac{1}{8}}^{8} \left(2x^{1/3} + x^{-2/3}\right)^2 dx$

$$= \pi \int_{\frac{1}{8}}^{8} \left(4x^{2/3} + 4x^{-1/3} + x^{-4/3}\right) dx$$

$$= \pi \left[\frac{12}{5}x^{5/3} + 6x^{2/3} - 3x^{-1/3}\right]_{\frac{1}{8}}^{8}$$

$$= \pi \left(\frac{993}{10} + \frac{177}{40}\right) = \frac{4149}{40}\pi$$

If $f(x)$ is only rotated θ around the x-axis

$\text{Volume}_\theta = \frac{\theta}{2\pi} \times \text{Vol}_{\text{fully rotated}}$

$\therefore \frac{461}{2} = \frac{\theta}{2\cancel{\pi}} \times \left(\frac{4149\cancel{\pi}}{40}\right)$

$\theta = \frac{40}{9}$ rad

10. The population of chimpanzees in a particular country consists of juveniles and adults. Juvenile chimpanzees do not reproduce.

In a study, the numbers of juvenile and adult chimpanzees were estimated at the start of each year. A model for the population satisfies the matrix system

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix} \qquad n = 0, 1, 2, \ldots$$

where a is a constant, and J_n and A_n are the respective numbers of juvenile and adult chimpanzees n years after the start of the study.

(a) Interpret the meaning of the constant a in the context of the model.

(1)

At the start of the study, the total number of chimpanzees in the country was estimated to be 64 000

According to the model, after one year the number of juvenile chimpanzees is 15 360 and the number of adult chimpanzees is 43 008

(b) (i) Find, in terms of a

$$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}$$

(3)

(ii) Hence, or otherwise, find the value of a.

(3)

(iii) Calculate the change in the number of juvenile chimpanzees in the first year of the study, according to this model.

(2)

Given that the number of juvenile chimpanzees is known to be in decline in the country,

(c) comment on the short-term suitability of this model.

(1)

A study of the population revealed that adult chimpanzees stop reproducing at the age of 40 years.

(d) Refine the matrix system for the model to reflect this information, giving a reason for your answer.

(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear.*)

(2)

10. a) $\begin{bmatrix} J_{n+1} \\ A_{n+1} \end{bmatrix} = \begin{bmatrix} a & 0.15 \\ 0.08 & 0.82 \end{bmatrix} \begin{bmatrix} J_n \\ A_n \end{bmatrix} = \begin{bmatrix} aJ_n + 0.15A_n \\ 0.08J_n + 0.82A_n \end{bmatrix}$

Question 10 continued

remain juvenile

adult chimpanzees that reproduce so new juvenile chimpanzees

$$J_{n+1} = aJ_n + 0.15A_n$$

$\therefore$ a represents the proportion of juvenile chimpanzees that survive & remain juveniles the next year.

b)(i) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a & 0.15 \\ 0.08 & 0.82 \end{bmatrix}^{-1} = \frac{1}{0.82a-0.012}\begin{bmatrix} 0.82 & -0.15 \\ -0.08 & a \end{bmatrix}$$

(ii) $$\begin{bmatrix} J_1 \\ A_1 \end{bmatrix} = \begin{bmatrix} a & 0.15 \\ 0.08 & 0.82 \end{bmatrix}\begin{bmatrix} J_0 \\ A_0 \end{bmatrix}$$

$$\begin{bmatrix} J_1 \\ A_1 \end{bmatrix} = \begin{bmatrix} 15360 \\ 43008 \end{bmatrix}$$

$$Pop_1 = M \times Pop_0$$

$$M^{-1} \times Pop_1 = \cancel{M^{-1}} \times \cancel{M} \times Pop_0$$

$$\begin{bmatrix} J_0 \\ A_0 \end{bmatrix} = \frac{1}{0.82a-0.012}\begin{bmatrix} 0.82 & -0.15 \\ -0.08 & a \end{bmatrix}\begin{bmatrix} 15360 \\ 43008 \end{bmatrix}$$

$$\begin{bmatrix} J_0 \\ A_0 \end{bmatrix} = \frac{1}{0.82a-0.012}\begin{bmatrix} 6144 \\ -1228.8+43008a \end{bmatrix}$$

We know total no. chimpanzees at the start (ie. J_0+A_0) $=64000$

$$\frac{1}{0.82a-0.012}(6144-1228.8+43008a) = 64000$$

$$4915.2+43008a = 52480a-768$$

$$a = \frac{3}{5}$$

Question 10 continued

(iii) $J_0 = \dfrac{6144}{0.82a - 0.012} = \dfrac{6144}{0.82\left(\frac{3}{5}\right) - 0.012} = 12800$

change in juvenile populations $= 15360 - 12800 = 2560$

c) Because the number of juveniles has increased, the model isn't initially predicting a decline
∴ not suitable in the short term

d) There must be a 3rd category for chimpanzees above 40 +, as these chimpanzees cannot reproduce

e.g. $\begin{bmatrix} J_{n+1} \\ A_{n+1} \\ M_{n+1} \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ 0.08 & c & 0 \\ 0 & d & e \end{bmatrix} \begin{bmatrix} J_n \\ A_n \\ M_n \end{bmatrix}$ none of the mature chimpanzees will reproduce, so $0M_n$

mature chimpanzees

P 5 8 3 3 3 A 0 3 0 3 2